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Vertices from replica in a random matrix theory

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Abstract

Kontsevitch's work on Airy matrix integrals has led to explicit results for the intersection numbers of the moduli space of curves. In a subsequent work Okounkov rederived these results from the edge behavior of a Gaussian matrix integral. In our work we consider the correlation functions of vertices in a Gaussian random matrix theory , with an external matrix source. We deal with operator products of the form $< \prod_{i=1}^{n} \frac{1}{N} \mathrm{tr} M^{k_i} >$, in a $\frac{1}{N}$ expansion. For large values of the powers k_i, in an appropriate scaling limit relating large k's to large N, universal scaling functions are derived. Furthermore we show that the replica method applied to characteristic polynomials of the random matrices, together with a duality exchanging N and the number of points, allows one to recover Kontsevich's results on the intersection numbers, through a simple saddle-point analysis.

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1 Introduction

Random matrix theory (RMT) has been applied to many physical problems, and also to mathematical subjects such as the distribution of zeros of Riemann zeta function or combinatorial problems and it has led to several meaningful results [1]. It also plays an essential role in the theory of random surfaces and for string theory. Several kinds of correlation functions in random matrix theory have been studied. In previous papers, we have studied the correlation function of the eigenvalues [4], and the correlations of the characteristic polynomials[7, 9], for which we have derived explicit integral representations.

In this article, we consider the correlation functions of vertices on the basis of previously derived integral representations. The diagrammatic representation of the vertex $\langle \mathrm{tr} M^k \rangle$, where M is a random matrix, is obtained through Wick's theorem, by the pairings of k-legs, each leg carrying the two indices (i, j) of the matrix element M_{ij}. For $N \times N$ matrices, the two indices run from 1 to N : $i, j = 1, ..., N$.

We restrict ourselves in this article to complex Hermitian random matrices. The distribution function for M is Gaussian with an external matrix source A.

$$P_A(M) = \frac{1}{Z_A} e^{-\frac{N}{2}\mathrm{tr}M^2 - N\mathrm{tr}MA} \tag{1}$$

When one sets $A = 0$, it reduces to the usual Gaussian unitary ensemble (GUE).

The correlation functions for the vertices $V(k_1, ..., k_n)$ are defined as

$$V(k_1, ..., k_n) = \frac{1}{N^n} < \mathrm{tr}M^{k_1}\mathrm{tr}M^{k_2} \cdots \mathrm{tr}M^{k_n} > \tag{2}$$

The normalization is chosen so that they have a finite large-N limit. These functions are closely related to the Fourier transform of the correlation functions of the eigenvalues,

$$U(t_1, ..., t_n) = \int_{-\infty}^{\infty} e^{i\sum t_i\lambda_i} R_n(\lambda_1, ...\lambda_n) \prod_1^N d\lambda_i \tag{3}$$

where the correlation function of the eigenvalues is

$$R_n(\lambda_1, ..., \lambda_n) =< \prod_{i=1}^{n} \frac{1}{N}\mathrm{tr}\delta(\lambda_i - M) > \tag{4}$$

Indeed

$$U(t_1, ..., t_n) =< \frac{1}{N^n} \prod_{i=1}^{n} \mathrm{tr}e^{it_iM} > \tag{5}$$

are generating functions of the $V(k)$ since

$$U(t_1,...,t_n) = \sum_{k_i=0}^{\infty} < \mathrm{tr}M^{k_1}\mathrm{tr}M^{k_2}\cdots\mathrm{tr}M^{k_n} > \frac{(it_1)^{k_1}\cdots(it_n)^{k_n}}{k_1!k_2!\cdots k_n!N^n} \tag{6}$$

When the distribution of the random matrix is Gaussian, the average of the vertices gives the numbers of pairwise gluing of the legs of the vertex operators. The dual cells of these vertices are polygons, whose edges are pairwise glued. We thereby generate orientable surfaces, which are discretized Riemann surfaces of given genus.

Okounkov and Pandharipande [16, 17] have shown that the intersection numbers, computed by Kontsevich [15], may be obtained by taking a simultaneous large N and large k_i limit. Furthermore the correlation functions of these vertices are interesting, since they give universal numbers in the large N limit.

We have investigated in an earlier work the F.T. of the n-point correlation function $U(t_1,...t_n)$ for the GUE, and found a simple contour integral representation valid even for finite N [3, 4].

In this article, we extend this integral representation to the vertex correlations $V(k_1,..k_n)$, and examine the scaling region for large k_i and large N. In this integral representation, the asymptotic evaluation by the saddle-point method requires a careful examination to deal with pole terms. This leads to a practical way to compute intersection numbers which we discuss in detail. We also show that the F.T. of the correlation functions (C.F.) of GUE near the edge point of the support of the asymptotic spectrum, is equivalent to Kontsevich's Airy matrix model ; the identification is based on the replica method and over a duality for computing averages of characteristic polynomials.

The article is organized as follows.

In section 2, we consider the F.T. of the one point correlation function at a bulk generic point in the large N limit. This is done by a contour integral representation, and we obtain the behavior of $< \mathrm{tr}M^{2k} >$ when N and k are large. We show that in this limit, one recovers the behavior of the one point function near the edge point of the spectrum.

In section 3, we consider the correlation function of two vertices.

In section 4, we investigate the correlations of the n-vertices.

In section 5, we introduce a replica method, relying on averages of characteristic polynomials. This, together with a duality, allows us to make connexion with the Kontsevich model, recovering thereby generating functions for the intersection numbers.

In section 6, we present a short summary.

2 One point correlation function

The correlation function $R_n(\lambda_1, ..., \lambda_n)$ defined by

$$R_n(\lambda_1, ..., \lambda_n) = < \prod_{i=1}^{n} \frac{1}{N} \mathrm{tr}(\delta(\lambda_i - M)) > . \tag{7}$$

is thus equal to

$$R_n(\lambda_1, ..., \lambda_n) = < \prod_{i=1}^{n} \frac{1}{N} \mathrm{tr}(\int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-it_i(\lambda_i - M)}) > . \tag{8}$$

The Fourier transform $U(t_1, ..., t_n)$ of $R_n(\lambda_1, ..., \lambda_n)$ is thus given by

$$\begin{aligned} U(t_1, ..., t_n) &= \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\lambda_i e^{i \sum_i t_i \lambda_i} R_n(\lambda_1, ..., \lambda_n) \qquad (9) \\ &= < \frac{1}{N^n} \prod_{i=1}^{n} \mathrm{tr} e^{it_i M} > . \qquad (10) \end{aligned}$$

Note that $U(t_1, ..., t_n)$ is normalized to one when all $t_i = 0$. The function $U(t_1, ..., t_n)$ is the generating function of the correlation $V(t_1, ..., t_n)$ as shown in (6).

These F.T. of the correlation functions $U(t_1, ..., t_n)$ were investigated in our earlier study of the kernels for the correlation functions [2, 3] ; there we had derived an exact integral representation for these correlation functions. We consider the probability distribution

$$P_A(M) = \frac{1}{Z_A} e^{-\frac{N}{2} \mathrm{tr} M^2 - N \mathrm{tr} MA}$$

then one finds the exact result

$$\begin{aligned} U(t_1, ..., t_n) &= \frac{1}{(t_1 \cdots t_n)} e^{-\frac{1}{2N}(t_1^2 + \cdots + t_n^2)} \\ &\times \oint \frac{du_1 \cdots du_n}{(2\pi i)^n} e^{i \sum_p t_p u_p} \prod_{p=1}^{n} \prod_{\gamma=1}^{N} (1 + \frac{it_p}{N(u - a_\gamma)}) \\ &\times \prod_{p<q}^{n} \frac{[u_p - u_q + \frac{i}{N}(t_p - t_q)](u_p - u_q)}{(u_p - u_q + \frac{i}{N} t_p)(u_p - u_q - \frac{i}{N} t_q)} \qquad (11) \end{aligned}$$

where the integration contours circle around the eigenvalues a_γ , $(\gamma = 1, \cdots N)$, of the source matrix A in the anticlockwise direction.

For the one-point function

$$U(t) = < \frac{1}{N} \mathrm{tr}\, e^{itM} > \tag{12}$$

the exact integral representation for finite N is thus [2]

$$U(t) = \frac{1}{it}\oint \frac{du}{2\pi i}\prod_{\gamma=1}^{N}\left(\frac{u-a_\gamma+\frac{it}{N}}{u-a_\gamma}\right)e^{-\frac{t^2}{2N}+itu}, \tag{13}$$

which reduces, for the pure Gaussian model $a_\gamma = 0$, to

$$U(t) = \frac{1}{it}\oint \frac{du}{2\pi i}\left(1+\frac{it}{Nu}\right)^N e^{-\frac{t^2}{2N}+itu}. \tag{14}$$

For this sourceless GUE one obtains immediately in the large N limit

$$U(t) = \frac{1}{it}\oint \frac{du}{2\pi i}e^{it(u+\frac{1}{u})}. \tag{15}$$

The generating function for Bessel functions $J_j(x)$,

$$\begin{aligned} e^{it(u+\frac{1}{u})} &= \sum_{j=-\infty}^{\infty}(iu)^j J_j(2t) \\ J_{-j}(x) &= (-1)^j J_j(x), \end{aligned} \tag{16}$$

leads to

$$U(t) = \frac{1}{t}J_1(2t). \tag{17}$$

The semi-circle law for the density of states of the GUE follows :

$$\begin{aligned} \rho(x) &= \int_{-\infty}^{\infty}\frac{dt}{2\pi}U(t)e^{-ixt} \\ &= 0, \qquad\qquad\qquad (\frac{x}{2}\geq 1) \\ &\quad \frac{1}{\pi}\sqrt{1-(\frac{x}{2})^2}, \qquad (\frac{x}{2}\leq 1). \end{aligned} \tag{18}$$

Returning now to the exact expression (14) for finite N one finds

$$\begin{aligned} U(t) &= \sum_{k=0}^{\infty}\frac{(it)^{2k}}{N^k}\left[\sum_{l=0}^{k}\frac{\Gamma(N)}{\Gamma(N-k+l)\Gamma(k-l+1)\Gamma(k-l+2)\Gamma(l+1)2^l}\right] \\ &= 1+\frac{(it)^2}{2}+\frac{1}{12}(1+\frac{1}{2N^2})(it)^4+\frac{1}{144}(1+\frac{2}{N^2})(it)^6+\cdots \end{aligned} \tag{19}$$

There are no odd powers of $\frac{1}{N}$ in this expansion, as is well-known for the GUE case, for which the successive terms of the $1/N$-expansion are of the form

$1/N^{2g}$, where g is the genus of the surface generated by the Wick contractions. From the relation between $U(t)$ and $< \mathrm{tr} M^{2k} >$, we obtain

$$
\begin{aligned}
& \frac{1}{N} < \mathrm{tr} M^{2k} >= \frac{(2k)!}{N^k}[\sum_{l=0}^{k} \frac{\Gamma(N)}{\Gamma(N-k+l)\Gamma(k-l+1)\Gamma(k-l+2)\Gamma(l+1)2^l}] \\
= & \frac{(2k)!}{N^k}[\frac{(N-1)(N-2)\cdots(N-k)}{k!(k+1)!} + \frac{(N-1)(N-2)\cdots(N-k+1)}{(k-1)!k!2} \\
+ & \frac{(N-1)(N-2)\cdots(N-k+2)}{(k-2)!(k-1)!8} + \cdots] \qquad (20)
\end{aligned}
$$

This exact representation leads to (20) the expansion

$$
\begin{aligned}
& \frac{1}{N} < \mathrm{tr} M^{2k} >= \frac{(2k)!}{k!(k+1)!}[1 + \frac{k(k-1)(k+1)}{12N^2} \\
& + \frac{k(k+1)(k-1)(k-2)(k-3)(5k-2)}{1440N^4} + O(\frac{1}{N^6})] \qquad (21)
\end{aligned}
$$

The large N limit is the first term of the above expansion :

$$
\lim_{N\to\infty} \frac{1}{N} < \mathrm{tr} M^{2k} >= \frac{(2k)!}{k!(k+1)!} \qquad (22)
$$

the k-th Catalan numbers. For large k this number behaves as $\frac{1}{\sqrt{\pi}}\frac{1}{k^{\frac{3}{2}}}4^k$. Therefore the resolvent

$$
\begin{aligned}
G(\lambda) &= < \frac{1}{N}\mathrm{tr}\frac{1}{\lambda - M} > \\
&= \frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{1}{\lambda^{2k}} < \mathrm{tr} M^{2k} > \qquad (23)
\end{aligned}
$$

has a square root singularity at $\lambda_c^2 = 4$. This corresponds of course to the vanishing of the asymptotic density of state (18) as a square root at the edge.

Returning to the $\frac{1}{N}$ expansion of $< \mathrm{trM^{2k}} >$ one finds for large k

$$
\begin{aligned}
& \frac{1}{N} < \mathrm{tr} M^{2k} >\sim \frac{1}{\sqrt{\pi}}\frac{1}{k^{\frac{3}{2}}}4^k(1 - \frac{21}{8k} + O(\frac{1}{k^2})) \\
& \times[1 + \frac{k(k^2-1)}{12N^2} + \frac{k(k+1)(k-1)(k-2)(k-3)(5k-2)}{1440N^4} + O(\frac{1}{N^6})]. \qquad (24)
\end{aligned}
$$

When k is of the order $k \sim N^{\frac{2}{3}}$, the above series exhibits a scaling behavior:

$$
\frac{1}{N} < \mathrm{tr} M^{2k} >\sim \frac{1}{\sqrt{\pi}k^{\frac{3}{2}}}4^k[1 + \frac{k^3}{12N^2} + \frac{k^6}{(12)^2 2! N^4} + O(\frac{1}{N^6})]. \qquad (25)
$$

The power 4^k corresponds to the location of the edge of the support of the asymptotic spectrum ($\lambda_c^2 = 4$), and it is not universal. But the successive terms being powers of k^3/N^2 is universal, since this feature is related to the square root vanishing of the density of states. The scaling function in this double limit of large N and large k when k behaves as $k \sim N^{2/3}$, is also universal. The above coefficients of the scaling function in (25) provide the intersection numbers of the moduli of curves. The universality of the coefficients of the series in powers of k^3/N^2 corresponds to the F.T. near the edge of the spectrum $\lambda \sim \lambda_c$, for the universal Airy kernel.

In order to obtain the full scaling function of k^3/N^2, and not simply the first terms of the expansion as in (25) , we now consider an exact integral representation for $< \mathrm{tr} M^{2k} >$. From (6)

$$\begin{aligned} \frac{1}{N} < \mathrm{tr} M^{2k} > &= (2k)! \oint \frac{dt}{2\pi i} \frac{U(t)}{i^{2k} t^{2k+1}} \\ &= \frac{(2k)!}{i(-1)^k} \oint \oint \frac{dtdu}{(2\pi i)^2} \frac{1}{t^{2k+2}} (1 + \frac{it}{Nu})^N e^{-\frac{t^2}{2N} + itu} \end{aligned} \quad (26)$$

For large k, large N, we apply the saddle-point method to (26). The integrand in (26) behaves as $e^{N\phi}$, with

$$\phi = -2k \ln t + \frac{it}{u} + itu \quad (27)$$

The saddle point equations,

$$\begin{aligned} \frac{\partial \phi}{\partial u} &= it[-\frac{1}{u^2} + 1] = 0 \\ \frac{\partial \phi}{\partial t} &= -\frac{2k}{t} + i(\frac{1}{u} + u) = 0 \end{aligned} \quad (28)$$

give as solutions $u = \pm 1, t = \mp ik$. Expanding around the saddle-point in a standard way one obtains in the large k, large N limit

$$\frac{1}{N} < \mathrm{trM^{2k}} > \sim \frac{\mathrm{(2k)! e^{2k}}}{\mathrm{k^{2k+1}}} \frac{1}{4\pi \mathrm{k}} \mathrm{e}^{\frac{\mathrm{k}^3}{12\mathrm{N}^2}} \sim \frac{4^{\mathrm{k}}}{\sqrt{\pi} \mathrm{k}^{\frac{3}{2}}} \mathrm{e}^{\frac{\mathrm{k}^3}{12\mathrm{N}^2}} \quad (29)$$

We have thus obtained the scaling function

$$f(\frac{k^3}{N}) = \exp[\frac{k^3}{12N}] \quad (30)$$

in accordance with the expansion found hereabove in (25).

Instead of the bulk spectrum, we now consider the F.T. of the one point correlation function near the edge point ($\lambda \sim \lambda_c$). We denote by $\tilde{U}(t)$ the F.T. of the one point correlation function near the edge of the spectrum, to

distinguish it from the bulk $U(t)$. We expect from the previous argument that this $\tilde{U}(t)$ becomes $< \mathrm{trM^{2k}} >$ if one puts $t = -ik$.

To explore the vicinity of the edge, it is convenient to introduce a trivial external matrix source whose eigenvalues are all $a_\gamma = -1$ in (13), and multiply e^{-it} in order to compensate for this uniform shift. Thereby the edge is now at the origin and

$$\begin{aligned}\tilde{U}(t) &= \frac{e^{-it}}{it}\oint \frac{du}{2\pi i}(1+\frac{it}{N(1+u)})^N e^{-\frac{t^2}{2N}+itu} \\ &= \frac{e^{-it}}{it}\oint \frac{du}{2\pi i} e^{Nln(1+\frac{it}{N(1+u)})} e^{-\frac{t^2}{2N}+itu} \end{aligned} \tag{31}$$

In the regime in which $t \sim N^{2/3}$ and $u \sim N^{-1/3}$ one may expand for small u up to order u^2, and the contour integral becomes a saddle point Gaussian integral. Note that the term itu is cancelled in the exponent. We have

$$\begin{aligned}\tilde{U}(t) &= \frac{1}{it} e^{\frac{(it)^3}{3N^2}} \int_{-\infty}^{\infty} \frac{du}{2\pi i} e^{itu^2 - \frac{1}{N}t^2 u} \\ &= \frac{1}{2\sqrt{\pi}(it)^{\frac{3}{2}}} e^{\frac{(it)^3}{12N^2}} \end{aligned} \tag{32}$$

We thus recover the scaling function $f = \exp[k^3/12N^2]$, when we put $t = -ik$ in (32).The difference between $\frac{1}{N} < \mathrm{tr} M^{2k} >$ and $\tilde{U}(t)$ when we replace it by k, is only the prefactor $\frac{1}{2}4^k$, . We have thus shown that the scaling function of (30) in $< \mathrm{tr} M^{2k} >$ for $k \sim N^{\frac{2}{3}}$,namely $\exp[\frac{k^3}{12N^2}]$, may also be obtained from the F.T. of the one point correlation function near the edge by setting $t = -ik$.

Changing the prefactor by multiplying by $1/t^{1/2}$, and defining $x = -it/2^{1/3}$, we obtain the generating function,

$$\begin{aligned}F(x) &= \frac{1}{x^2} e^{\frac{x^3}{24N^2}} \\ &= \frac{1}{x^2} + \frac{x}{24N^2} + \frac{x^4}{(24)^2 2! N^4} + \cdots \\ &= \sum_{g=0}^{\infty} < \tau_{3g-2} > \frac{1}{N^{2g}} x^{3g-2}. \end{aligned} \tag{33}$$

The numbers $< \tau_k >$ in this expansion coincide with the intersection number of the moduli of curves, as we will be justified below by the replica method.

From the expansion (33), we obtain $< \tau_j >$ as

$$< \tau_{3g-2} >_g = \frac{1}{(24)^g g!} \quad (g = 0, 1, 2, ...). \tag{34}$$

These numbers agree with the values of the intersection numbers computed earlier by Kontsevitch and Witten [15, 19]. For $< \tau_0 >$, we need a special consideration in (33). If we put $g = \frac{2}{3}$, we get $< \tau_0 >$. However by definition $< \frac{1}{N}\mathrm{tr}M^{2k} >= 1$ for $k = 0$. Therefore we define $< \tau_0 >_{g=0}= 1$ instead of $< \tau_{-2} >_{g=0}= 1$ for comparison with the intersection numbers. We will discuss these intersection numbers later.

We have used here the integral representation to derive $\tilde{U}(t)$. Since this is related to the edge problem, we could have used instead the Airy kernel $K_A(\lambda, \lambda)$,

$$\begin{aligned} K_A(\lambda, \mu) &= \frac{A_i'(\lambda)A_i(\mu) - A_i(\lambda)A_i'(\mu)}{\lambda - \mu} \\ &= \int_0^\infty A_i(\lambda + z)A_i(\mu + z)dz \end{aligned} \tag{35}$$

where the Airy function $A_i(x)$ is given by

$$A_i(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{i}{3}\xi^3 + ix\xi}d\xi. \tag{36}$$

Near the edge of the spectrum $\lambda = 2$, the density of state is given by $K_A(\lambda, \lambda)$ in the appropriate large N scaling. Let us verify that one can recover the previous result from there :

$$\begin{aligned} \tilde{U}(t) &= \int_{-\infty}^{\infty} d\lambda e^{it\lambda}\int_0^\infty dz A_i^2(\lambda + z) \\ &= \frac{1}{(2\pi)^2}\int_0^\infty dz\int_{-\infty}^{\infty} e^{it\lambda}\int_{-\infty}^{\infty} e^{\frac{i}{3}\xi^3 + i\xi(\lambda+z)}d\xi\int_{-\infty}^{\infty} e^{\frac{i}{3}\eta^3 + i(-t+z)\eta}d\eta \\ &= \frac{1}{2\pi}\int_0^\infty dz\int_{-\infty}^{\infty} d\xi\int_{-\infty}^{\infty} d\eta\delta(\xi + \eta + t)e^{\frac{i}{3}\xi^3 + \frac{i}{3}\eta^3 + iz(\xi+\eta)} \\ &= \int_0^\infty dz\frac{1}{2\sqrt{i\pi t}}e^{\frac{(it)^3}{12} - izt} \\ &= \frac{1}{2\sqrt{\pi}}\frac{1}{(it)^{3/2}}e^{\frac{1}{12}(it)^3} \end{aligned} \tag{37}$$

which coincides as expected with (32).

3 Two-point correlation function

In the case of the two-point correlation function, we have

$$R_2(\lambda_1, \lambda_2) =< \frac{1}{N}\mathrm{tr}\delta(\lambda_1 - M)\frac{1}{N}\mathrm{tr}\delta(\lambda_2 - M) > \tag{38}$$

and the F.T. of $R_2(\lambda_1, \lambda_2)$ is

$$U(t_1, t_2) =< \frac{1}{N}\mathrm{tr}e^{it_1 M}\frac{1}{N}\mathrm{tr}e^{it_2 M} > \tag{39}$$

This correlation function has been obtained in closed form, for finite N, with the help of the HarishChandra-Itzykson-Zuber integral, in [3]

$$\begin{aligned} U(t_1, t_2) &= \sum_{\alpha_1,\alpha_2} \frac{\prod_{i<j}[a_i - a_j + \frac{i}{N}t_1(\delta_{i,\alpha_1} - \delta_{j,\alpha_1}) + \frac{i}{N}t_2(\delta_{i,\alpha_2} - \delta_{j,\alpha_2})]}{\prod_{i<j}(a_i - a_j)} \\ &\times \ e^{it_1 a_{\alpha_1} + it_2 a_{\alpha_2} - \frac{1}{2N}t_1^2 - \frac{1}{2N}t_2^2 - \frac{1}{N}t_1 t_2 \delta_{\alpha_1,\alpha_2}} \end{aligned} \tag{40}$$

This sum is then divided in two parts; $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$. The first part gives

$$U^I(t_1, t_2) = \sum_{\alpha_1}\prod_{i<j} \frac{[a_i - a_j + \frac{i}{N}(t_1 + t_2)(\delta_{i,\alpha_1} - \delta_{j,\alpha_1})]}{(a_i - a_j)} e^{-\frac{1}{2N}(t_1+t_2)^2 + i(t_1+t_2)a_{\alpha_1}} \tag{41}$$

This may be expressed as the contour integral

$$U^I(t_1, t_2) = \frac{1}{i(t_1 + t_2)}\oint \frac{du}{2\pi i}\prod_{\gamma=1}^{N}[1 + \frac{i(t_1 + t_2)}{N(u - a_\gamma)}]e^{i(t_1+t_2)u + \frac{1}{2N}((it_1)^2 + (it_2)^2)} \tag{42}$$

which is nothing but $U(t_1 + t_2)$ from (13).

The second term is expressed by the double contour integral

$$\begin{aligned} U^{II}(t_1, t_2) &= e^{-\frac{1}{2N}(t_1^2 + t_2^2)}\oint \frac{du_1 du_2}{(2\pi i)^2}e^{it_1 u_1 + it_2 u_2}\prod_{\gamma=1}^{N}(1 + \frac{it_1}{N(u_1 - a_\gamma)})(1 + \frac{it_2}{N(u_2 - a_\gamma)}) \\ &\times \frac{1}{t_1 t_2}\frac{(u_1 - u_2 + \frac{1}{N}(it_1 - it_2))(u_1 - u_2)}{(u_1 - u_2 + \frac{i}{N}t_1)(u_1 - u_2 - \frac{i}{N}t_2)} \end{aligned} \tag{43}$$

Noting that

$$1 - \frac{t_1 t_2}{N^2(u_1 - u_2 + \frac{i}{N}t_1)(u_1 - u_2 - \frac{1}{N}it_2)} = \frac{(u_1 - u_2 + \frac{1}{N}(it_1 - it_2))(u_1 - u_2)}{(u_1 - u_2 + \frac{i}{N}t_1)(u_1 - u_2 - \frac{i}{N}t_2)} \tag{44}$$

we find U^{II} is a sum of a disconnected part and a connected part.

Therefore, we find the connected part of $U^{II}(t_1, t_2)$ as the contour integral

$$\begin{aligned} U_c^{II}(t_1, t_2) &= -e^{\frac{1}{2N}((it_1)^2 + (it_2)^2)}\oint \frac{du_1 du_2}{(2\pi i)^2}e^{it_1 u_1 + it_2 u_2}(1 + \frac{it_1}{Nu_1})^N(1 + \frac{it_2}{Nu_2})^N \\ &\times \ \frac{1}{N^2(u_1 - u_2 + \frac{it_1}{N})(u_1 - u_2 - \frac{it_2}{N})} \end{aligned} \tag{45}$$

We have set here all the $a_\gamma = 0$ to deal with the pure GUE. The contour is around $u_1, u_2 = 0$. This $U_c(t_1, t_2)$ may be expanded in powers of t_1 and t_2. Together with the exponential factor, it yields the expansion

$$U_c^{II}(t_1,t_2) = \frac{1}{N}[1 - \frac{1}{2}(t_1+t_2)^2 + \frac{1}{N}t_1t_2 + \frac{1}{12}(1+\frac{1}{2N^2})(t_1^4+t_2^4)$$
$$+(\frac{1}{3} - \frac{1}{2N} + \frac{1}{6N^2})(t_1^3t_2 + t_1t_2^3) + (\frac{1}{2} - \frac{1}{2N} + \frac{1}{4N^2})t_1^2t_2^2 + O(t^6)] \tag{46}$$

One may wonder why we have obtained odd powers of $\frac{1}{N}$ in this expression but one can check that combined with $U(t_1+t_2)$, the odd power in $\frac{1}{N}$ of $U_c^{II}(t_1,t_2)$ cancels.

If we compute the integral representation of $U_c^{II}(t_1,t_2)$ by deforming the contours to collect the contributions of the poles at $u_1 = \infty$ and $u_2 = 0$, instead of $u_1 = 0, u_2 = 0$, we obtain

$$U_c(t_1,t_2)_{(\infty,0)} = \frac{1}{N}[t_1t_2 - \frac{1}{2}t_1^2t_2^2 - \frac{1}{2}(t_1^3t_2 + t_1t_2^3) + O(t^6)] \tag{47}$$

From the expression of $U(t)$ and $U_c^{II}(t_1,t_2)$ found in (19) and (46), we have

$$\frac{1}{N}U(t_1+t_2) - U_c^{II}(t_1,t_2) = -\frac{1}{N}[t_1t_2 - \frac{1}{2}(t_1^3t_2 + t_1t_2^3) - \frac{1}{2}t_1^2t_2^2 + O(t^6)] \tag{48}$$

which is indeed equal to $-U_c(t_1,t_2)_{(\infty,0)}$ in (47) and justifies the deformation of the contour integration in order to collect the residue of the pole at $u_1 = \infty$.

Let us first consider the connected part of the two-point correlation function in the large N limit.

$$U_c(t_1,t_2) = -\frac{1}{N^2}\oint \frac{du_1du_2}{(2\pi i)^2} e^{it_1u_1 + it_2u_2 + \frac{it_1}{u_1} + \frac{it_2}{u_2}} \frac{1}{u_1^2}[\sum_{n=0}^{\infty}(\frac{u_2}{u_1})^n]^2 \tag{49}$$

Using the Bessel function formula (16), we find from the residues at $u_1 = \infty$ and $u_2 = 0$,

$$U_c(t_1,t_2) = \frac{1}{N^2}\sum_{l=0}^{\infty}(-1)^l(l+1)J_{l+1}(2t_1)J_{l+1}(2t_2) \tag{50}$$

Noting that the F.T. of the Bessel functions $J_{l+1}(2t)$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} dt J_{l+1}(2t)e^{it\lambda} &= 2(4-\lambda^2)^{-\frac{1}{2}}\cos[(l+1)\phi] \quad ((l+1)\ \ is\ \ even) \\
\int_{-\infty}^{\infty} dt J_{l+1}(2t)e^{it\lambda} &= 2i(4-\lambda^2)^{-\frac{1}{2}}\sin[(l+1)\phi] \quad ((l+1)\ \ is\ \ odd) \\
\sin\phi &= \frac{\lambda}{2}.
\end{aligned} \tag{51}$$

we obtain by inverse F.T.

$$
\begin{aligned}
R_2(\lambda_1,\lambda_2) &= \frac{1}{N^2(2\pi i)^2}\int dt_1 dt_2 e^{i\lambda_1 t_1 + i\lambda_2 t_2}\sum_{l=0}^{\infty}(-1)^l (l+1) J_{l+1}(2t_1)J_{l+1}(2t_2)\\
&= \frac{1}{N^2(2\pi)^2}\sum_{l=0}^{\infty}(-1)^l(l+1)[e^{i(l+1)\phi_1}+(-1)^{l+1}e^{-i(l+1)\phi_1}]\\
&\quad \times [e^{i(l+1)\phi_2}+(-1)^{l+1}e^{-i(l+1)\phi_2}]\frac{1}{\sqrt{(4-\lambda_1^2)(4-\lambda_2^2)}}\\
&= -\frac{1}{2N^2\pi^2}\frac{1}{(\lambda_1-\lambda_2)^2}\frac{4-\lambda_1\lambda_2}{\sqrt{(4-\lambda_1^2)(4-\lambda_2^2)}}. \qquad (52)
\end{aligned}
$$

This result agrees with the earlier derivation of [10].

We now consider the integral representation for $< \mathrm{tr}M^{k_1}\mathrm{tr}M^{k_2} >$. We have

$$
\frac{1}{N^2} < \mathrm{tr}M^{2k_1}\mathrm{tr}M^{2k_2} >_c = \frac{(2k_1)!(2k_2)!}{(-1)^{k_1+k_2}}\oint\oint \frac{dt_1 dt_2}{(2\pi i)^2}\frac{U_c(t_1,t_2)}{t_1^{2k_1+1}t_2^{2k_2+1}} \qquad (53)
$$

where $U_c(t_1,t_2) = U_c^{II}(t_1,t_2) - U(t_1+t_2)$.

$$
\begin{aligned}
\frac{< \mathrm{tr}M^{2k_1}\mathrm{tr}M^{2k_2} >_c^{II}}{N^2} &= -\frac{(2k_1)!(2k_2)!}{N^2(-1)^{k_1+k_2}}\oint\frac{dt_1dt_2}{(2\pi i)^2}\oint\frac{du_1du_2}{(2\pi i)^2}\frac{e^{-\frac{1}{2N}(t_1^2+t_2^2)+it_1u_1+it_2u_2}}{t_1^{2k_1+1}t_2^{2k_2+1}}\\
&\quad\times\frac{(1+\frac{it_1}{Nu_1})^N(1+\frac{it_2}{Nu_2})^N}{(u_1-u_2+\frac{it_1}{N})(u_1-u_2-\frac{it_2}{N})} \qquad (54)
\end{aligned}
$$

We are interested in the large N and large k_1, k_2 behavior, but in the region in which the k_i are of order $N^{2/3}$. As for the previous calculation of $\frac{1}{N} < \mathrm{tr}M^{2k} >$, after exponentiation , we find again that the saddle points are $t_{1c} = -ik_1$ and $t_{2c} = -ik_2$ with $u_1 = u_2 = 1$ in (54). Then we expand t_1 and t_2 near the saddle-points

$$
\begin{aligned}
t_1 &= -ik_1(1+v_1)\\
t_2 &= -ik_2(1+v_2) \qquad (55)
\end{aligned}
$$

and expand for v_1, v_2 small. The integrations over v_1, v_2 become Gaussian, and they are equivalent to the replacement of t_i by their saddle point values $-ik_i$ (i=1,2). Therefore we have for large k and N, after the shift $u_i \to 1+u_i$,

$$
\begin{aligned}
&\frac{1}{N^2} < \mathrm{tr}M^{2k_1}\mathrm{tr}M^{2k_2} >_c^{II}\\
&= -C(k_1,k_2)e^{\frac{1}{3N^2}(k_1^3+k_2^3)}\int_{-\infty}^{\infty}\frac{du_1du_2}{(2\pi i)^2}\frac{e^{k_1u_1^2+\frac{k_1^2}{N}u_1+k_2u_2^2+\frac{k_2^2}{N}u_2}}{(u_1-u_2+\frac{k_1}{N})(u_1-u_2-\frac{k_2}{N})} \qquad (56)
\end{aligned}
$$

where the constant $C(k_1,k_2)$ is

$$
\begin{aligned}
C(k_1,k_2) &= \frac{(2k_1)!(2k_2)!e^{2k_1+2k_2}}{(-1)^{k_1+k_2}\sqrt{k_1k_2}k_1^{2k_1}k_2^{2k_2}} \\
&\sim \frac{1}{4^{2k_1+2k_2}4\pi k_1k_2}
\end{aligned}
\tag{57}
$$

The integration in (56) requires a careful examination of the pole terms. Since the denominator may vanish, we use

$$
\frac{1}{u_1-u_2+\frac{k_1}{N}-i\epsilon} = P(\frac{1}{u_1-u_2+\frac{k_1}{N}}) + i\pi\delta(u_1-u_2+\frac{k_1}{N}) \tag{58}
$$

The δ function contibution is nothing but $\frac{1}{N^2} < \mathrm{trM}^{2k_1+2k_2} >$. The principal part is evaluated by writing the denominator as

$$
\frac{1}{u_1-u_2+\frac{k_1}{N}} = -i\int_0^\infty d\alpha e^{i(u_1-u_2+\frac{k_1}{N})\alpha} \qquad (Imk_1>0) \tag{59}
$$

The integration in (56) becomes

$$
\begin{aligned}
I_2 &= \int_{-\infty}^{\infty}\frac{du_1du_2}{(2\pi i)^2}\int_0^\infty d\alpha d\beta e^{\sum_{i=1}^2(k_iu_i^2+\frac{k_i^2}{N}u_i)+i\alpha(u_1-u_2+\frac{k_1}{N})+i\beta(u_1-u_2-\frac{k_2}{N})} \\
&= \frac{1}{4\pi\sqrt{k_1k_2}}\int_0^\infty d\alpha d\beta e^{-k_1(\frac{k_1}{2N}+\frac{i(\alpha+\beta)}{2k_1})^2-k_2(\frac{k_2}{2N}-\frac{i(\alpha+\beta)}{2k_2})^2+\frac{i\alpha k_1}{N}-\frac{i\beta k_2}{N}}
\end{aligned}
\tag{60}
$$

In this representation we have assumed that $Imk_1>0, Imk_2<0$, but after integration over α,β, this condition becomes irrelevant. We replace

$$
\alpha+\beta = x, \quad \beta = z \tag{61}
$$

with

$$
0<z<x \tag{62}
$$

Thus we write I_2 as

$$
\begin{aligned}
I_2 &= \frac{e^{-\frac{k_1^3+k_2^3}{4N^2}}}{4\pi\sqrt{k_1k_2}}\int_0^\infty dx\int_0^x dz e^{\frac{k_1+k_2}{4k_1k_2}x^2+\frac{i(k_1+k_2)}{2N}x-\frac{i(k_1+k_2)}{N}z} \\
&= \frac{iN}{4\pi\sqrt{k_1k_2}(k_1+k_2)}e^{-\frac{k_1^3+k_2^3}{4N^2}+\frac{k_1k_2(k_1+k_2)}{4N^2}} \\
&\times\int_{-\frac{ik_1k_2}{N}}^{\frac{ik_1k_2}{N}} dx e^{\frac{k_1+k_2}{4k_1k_2}x^2}
\end{aligned}
\tag{63}
$$

After mutiplication by $e^{\frac{k_1^3+k_2^3}{3N^2}}$, we obtain

$$e^{\frac{1}{3N^2}(k_1^3+k_2^3)}I_2 = -\frac{\sqrt{k_1k_2}}{2\pi(k_1+k_2)}e^{\frac{(k_1+k_2)^3}{12N^2}}\sum_{l=0}^{\infty}\frac{1}{l!(2l+1)4^lN^{2l}}(k_1k_2(k_1+k_2))^l \tag{64}$$

Finally

$$\frac{1}{N^2} < \mathrm{tr}M^{2k_1}\mathrm{tr}M^{2k_2} >_c = -C(k_1,k_2)\frac{1}{2\pi}\frac{\sqrt{k_1k_2}}{k_1+k_2}e^{\frac{1}{12}(k_1+k_2)^3}\sum_{l=0}^{\infty}\frac{[-k_1k_2(k_1+k_2)]^l}{l!(2l+1)(4N^2)^l} \tag{65}$$

Note that in this evaluation, we have already subtracted the δ-function term, when we have represented the pole terms in terms of integrals over α and β.

For making contact with Kontsevitch normalization [15] , we change k_1, k_2 to $\frac{k_1}{2^{1/3}}, \frac{k_2}{2^{1/3}}$, and multiply a factor $\frac{2\pi}{\sqrt{k_1k_2}}$. We obtain the intersection numbers for two points (n=2) as an expansion of the error function.

$$F(x_1,x_2) = \frac{1}{x_1+x_2}e^{\frac{(x_1+x_2)^3}{24N^2}}\sum_{l=0}^{\infty}\frac{[x_1x_2(x_1+x_2)]^l(-1)^l}{[8N^2]^l(2l+1)l!} \tag{66}$$

where we set $x_i = k_i$ within the appropriate factors. This function $F(x_1,x_2)$ is a generating function of the intersection numbers. If we expand it in powers of x_1 and x_2

$$F(x_1,x_2) = \sum_{l_1,l_2} < \tau_{l_1}\tau_{l_2} >_g \frac{x_1^{l_1}x_2^{l_2}}{N^{2g}} \tag{67}$$

the coefficient $< \tau_{l_1}\tau_{l_2} >$ is the intersection number for genus g ; the genus is specified by $3g-1 = l_1+l_2$, and then the coefficient is then given by returning to (66).

As seen previously for $< \mathrm{tr}M^{2k} >$, the asymptotic evaluation of $< \mathrm{tr}M^{2k_1}\mathrm{tr}M^{2k_2} >$ is given by the Fourier transform $U(t_1,t_2)$ near the end point.

From the above expression we obtain coefficients, which are the intersection numbers for n=2,

$$\begin{gathered} < \tau_2\tau_0 >_{g=1} = < \tau_0\tau_2 >_{g=1} = \frac{1}{24}, \quad < \tau_5\tau_0 >_{g=2} = < \tau_0\tau_5 >_{g=2} = \frac{1}{(24)^2 2!}, \\ < \tau_1^2 >_{g=1} = \frac{1}{24}, \quad < \tau_4\tau_1 >_{g=2} = \frac{1}{384}, \quad < \tau_3\tau_2 >_{g=2} = \frac{29}{5760} \end{gathered} \tag{68}$$

These numbers agree with Witten's earlier results [19]. For $< \tau_0^2 >_{g=0}$, we use the normalization, $< \tau_0^2 >= 1$, in analogy with $< \tau_0 >= 1$.

It is worth noticing that when we set $x_2 = 0$, i.e. $k_2 = 0$ in (65), we do obtain

$$F(x_1,0) = F(x_1) = \frac{1}{x_1}e^{\frac{x_1^3}{24N^2}}, \quad < \tau_{l_1}\tau_0 >=< \tau_{l_1} > . \tag{69}$$

The above relation is the string equation, as will be explained later.

4 The n-point correlations

The Fourier transform $U(t_1, ..., t_n)$ is given by (11). Using Cauchy determinant formula, it is expressed as a determinant. For the connected part, we take the longest cyclic rings for the indices p, q.

$$\begin{aligned} U_c(t_1, ..., t_n) &= \frac{1}{N^n} e^{-\frac{1}{2N}(t_1^2 + \cdots + t_n^2)} \\ &\times \oint \frac{du_1 \cdots du_n}{(2\pi i)^n} e^{i\sum_p t_p u_p} \prod_{p=1}^{n} \prod_{\gamma=1}^{N} (1 + \frac{it_p}{N(u_p - a_\gamma)}) \\ &\times \prod_{cycle} \frac{1}{\frac{i}{N}t_p + u_p - u_q}, \end{aligned} \tag{70}$$

where the last product is the maximal cycle for the indices (p,q). For instance, in the case n=3, we have two longest cycles $(1 \to 2 \to 3 \to 1)$ and $(1 \to 3 \to 2 \to 1)$,

$$\begin{aligned} &\prod_{cycle} \frac{1}{\frac{i}{N}t_p + u_p - u_q} = \frac{1}{(u_1 - u_2 + \frac{it_1}{N})(u_2 - u_3 + \frac{it_2}{N})(u_3 - u_1 + \frac{it_3}{N})} \\ &+ \frac{1}{(u_1 - u_3 + \frac{it_1}{N})(u_2 - u_1 + \frac{it_2}{N})(u_3 - u_2 + \frac{it_3}{N})} \end{aligned} \tag{71}$$

These two terms contribute to the connected part of $U(t_1, t_2, t_3)$ as

$$\begin{aligned} &U_c(t_1, t_2, t_3) = \frac{1}{N^3} e^{-\frac{1}{2N}(t_1^2 + t_2^2 + t_3^2)} \\ &\times \oint \frac{du_1 du_2 du_3}{(2\pi i)^3} e^{i\sum t_j u_j} [\frac{(1 + \frac{it_1}{Nu_1})^N (1 + \frac{it_2}{Nu_2})^N (1 + \frac{it_3}{Nu_3})^N}{(u_1 - u_2 + \frac{it_1}{N})(u_2 - u_3 + \frac{it_2}{N})(u_3 - u_1 + \frac{it_3}{N})} \\ &+ \frac{(1 + \frac{it_1}{Nu_1})^N (1 + \frac{it_2}{Nu_2})^N (1 + \frac{it_3}{Nu_3})^N}{(u_1 - u_3 + \frac{it_1}{N})(u_2 - u_1 + \frac{it_2}{N})(u_3 - u_2 + \frac{it_3}{N})}] \end{aligned} \tag{72}$$

We find that the two terms are identical, and they are symmetric polynomials of the t_p.

As we have seen, we have to add several terms to obtain the connected part of $U(t_1, ..., t_n)$, to deal with the cases $\alpha_i = \alpha_j$ in the summation implied by (40). The evaluation of this integral is an extenstion of the previous study of I_2.

For the n=3 case, we have in the large N limit, neglecting all $\frac{1}{N}$ terms,

$$I = \oint \frac{du_1 du_2 du_3}{(2\pi i)^3} \frac{e^{\sum_i it_i u_i + \frac{it_i}{u_i}}}{(u_1 - u_2)(u_2 - u_3)(u_3 - u_1)}$$

$$
\begin{aligned}
&= -\oint \frac{du_i}{(2\pi i)^3}\frac{1}{u_1^2 u_2}\sum_{l_1,l_2,l_3}(\frac{u_2}{u_1})^{l_1}(\frac{u_3}{u_1})^{l_3}(\frac{u_3}{u_2})^{l_2} \\
&\quad \times \sum_{s_1,s_2,s_3=-\infty}^{\infty}(iu_1)^{s_1}(iu_2)^{s_2}(iu_3)^{s_3}J_{s_1}(2t_1)J_{s_2}(2t_2)J_{s_3}(2t_3) \\
&= \sum_{l_i,m_j=0}^{\infty}\frac{(-1)^{m_1+m_2+m_3}t_1^{2m_1+l_1+l_3+1}t_2^{2m_2+l_1-l_2}t_3^{2m_3+l_2+l_3+1}}{m_1!(m_1+1+l_1+l_3)!m_2!(m_2+l_1-l_2)!m_3!(m_3+l_2+l_3+1)!}
\end{aligned}
\tag{73}
$$

We consider the coefficients of $(-1)^{k_1+k_2+k_3}t_1^{2k_1}t_2^{2k_2}t_3^{2k_3}$ of I, which is denoted by $I^{2k_1,2k_2,2k_3}$. We put

$$
\begin{aligned}
m_1 &= k_1-\frac{1}{2}-\frac{l_1}{2}-\frac{l_3}{2} \\
m_2 &= k_2-\frac{l_1}{2}+\frac{l_2}{2} \\
m_3 &= k_3-\frac{l_2}{2}-\frac{l_3}{2}-\frac{1}{2}
\end{aligned}
\tag{74}
$$

If l_1 is even, then l_2 is even and l_3 is odd. If l_1 is odd, then l_2 is odd and l_3 is even. These two cases give the same result, and give a factor 2 for l_1 even.We change $l_1 \to 2l_1$, $l_2 \to 2l_2$ and $l_3 \to 2l_3+1$.

$$
\begin{aligned}
&I^{2k_1,2k_2,2k_3} \\
&= \sum_{l_1,l_2,l_3=0}^{\infty}\frac{1}{(k_1-l_1-l_3-1)!(k_1+l_1+l_3+1)!(k_2-l_1+l_2)!(k_2+l_1-l_2)!} \\
&\times\frac{1}{(k_3-l_2-l_3-1)!(k_3+l_2+l_3+1)!}
\end{aligned}
\tag{75}
$$

This sum is expressed by the contour integration,

$$
\begin{aligned}
&I^{2k_1,2k_2,2k_3} \\
&= \frac{1}{(2k_1)!(2k_2)!(2k_3)!}\oint\frac{dxdydz}{(2\pi i)^3}\frac{(1+x)^{2k_1}(1+y)^{2k_2}(1+z)^{2k_3}}{x^{k_1}y^{k_2+1}z^{k_3}(1-xy)(1-xz)(1-\frac{z}{y})} \\
&= \frac{1}{(2k_1)!(2k_2)!(2k_3)!}\oint\frac{dxdydz}{(2\pi i)^3}\frac{(x+y)^{2k_1}(1+y)^{2k_2}(1+zy)^{2k_3}}{x^{k_1}y^{k_1+k_2+k_3+1}z^{k_3}(1-x)(1-z)(1-xz)}
\end{aligned}
\tag{76}
$$

where the contours are around $x=y=z=0$, and in the last line, we have made the change of variables $x \to \frac{x}{y}$ and $z \to zy$. When k_1, k_2 and k_3 are large, the saddle point for this integrals are $x_c = y_c = z_c = 1$; however the denominator vanishes at this point. Therefore, we first deform

the contours. The integral of (76) is invariant, except for a sign, under the change of variables, $x \to \frac{1}{x}$,$y \to \frac{1}{y}$ and $z \to \frac{1}{z}$, which transforms the contour around $x = y = z = 0$ into a contour at $x = y = z = \infty$. By Cauchy theorem, the sum of all residues has to vanish if we include the residues at infinity, Thereby we obtain the following identity between the different contour integrals for (76).

$$\oint_{x=y=z=0} F = -\frac{1}{2}[\oint_{x=1,y=0,z=0} + \oint_{z=1,x=0,y=0} + \oint_{x=\frac{1}{z},y=0,z=0}]F \tag{77}$$

where F is the integrand of (76). This identity is derived from the invariance, except for the overall sign, under the change $x \to \frac{1}{x}, y \to \frac{1}{y}, z \to \frac{1}{z}$ for the expression (76). We have

$$\begin{aligned}
&\oint_{x=y=z=0} \frac{dxdydz}{(2\pi i)^3} \frac{(x+y)^{2k_1}(1+y)^{2k_2}(1+zy)^{2k_3}}{x^{k_1}y^{k_1+k_2+k_3+1}z^{k_3}(1-x)(1-z)(1-xz)} \\
&= -\oint_{x=y=z=\infty} \frac{dxdydz}{(2\pi i)^3} \frac{(x+y)^{2k_1}(1+y)^{2k_2}(1+zy)^{2k_3}}{x^{k_1}y^{k_1+k_2+k_3+1}z^{k_3}(1-x)(1-z)(1-xz)}
\end{aligned} \tag{78}$$

Therefore, we obtain the identity (77) with the factor 1/2.

The double pole $\frac{1}{(1-z)^2}$, which appears for the contour integral around $x = 1$, is transformed into a single pole by an integration by parts over x, and the sigularity at $z = 1$ is cancelled by the numerator. Therefore we may now use the saddle point at $z_c = 1$, and obtain the large k_i behavior,

$$\begin{aligned}
I^{2k_1,2k_2,2k_3} &= \frac{4^{k_1+k_2+k_3}}{4\pi(2k_1)!(2k_2)!(2k_3)!(k_1+k_2+k_3)}[\sqrt{k_3(k_1+k_2)} \\
&+ \sqrt{k_1(k_2+k_3)} - \sqrt{k_2(k_1+k_3)}]
\end{aligned} \tag{79}$$

In order to appreciate the asymptotic behavior of these integrals, we have compared the exact value for the contour integral (76), multiplied by a factor $(2k_1)!(2k_2)!(2k_3)!$, to the large k estimates. For instance for $k_1 = 30, k_2 = 15, k_3 = 50$ the exact integral is 7.4632×10^{55}, and the large k_i asymptotic formula (79) gives instead 7.4864×10^{55}.

We have considered the region $u_1 \geq u_2 \geq u_3$. There are other regions $u_i \geq u_j \geq u_l$. Adding their contributions amounts to summing over permutations of the k_1, k_2 and k_3 ; taking also into account the I^{k_1,k_2,k_3} for l_1 odd, we obtain

$$\begin{aligned}
I^{k_1,k_2,k_3} &= \frac{4^{k_1+k_2+k_3}}{2\pi(2k_1)!(2k_2)!(2k_3)!(k_1+k_2+k_3)}[\sqrt{k_3(k_1+k_2)} + \sqrt{k_1(k_2+k_3)} \\
&+\sqrt{k_2(k_1+k_3)}]
\end{aligned} \tag{80}$$

This leads to,

$$\frac{1}{N^3} < \mathrm{tr} M^{2k_1} \mathrm{tr} M^{2k_2} \mathrm{tr} M^{2k_3} >_c$$
$$= \frac{4^{k_1+k_2+k_3}}{2\pi(k_1+k_2+k_3)}[\sqrt{k_3(k_1+k_2)} + \sqrt{k_1(k_2+k_3)} + \sqrt{k_2(k_1+k_3)}] \tag{81}$$

We have evaluated the leading term of order one in the large N limit of the three-point correlation function. However, this leading term is cancelled when we consider connected correlation functions. Indeed let us consider the expansion of $U(t)$ and $U(t_1, t_2)$:

$$\begin{aligned} U(t) &= \frac{1}{N}\oint \frac{du}{2\pi i}\frac{N}{it} e^{itu}(1+\frac{it}{Nu})^N e^{-\frac{t^2}{2N}} \\ &= 1 - \frac{1}{2}t^2 + \frac{1}{12}t^4 + \frac{1}{24N^2}t^4 + O(t^6) \end{aligned} \tag{82}$$

$$U_c(t_1,t_2) = \frac{1}{N^2} e^{-\frac{1}{2N}(t_1^2+t_2^2)} \oint \frac{du_1 du_2}{(2\pi i)^2} \frac{e^{it_1u_1+it_2u_2}(1+\frac{it_1}{Nu_1})^N(1+\frac{it_2}{Nu_2})^N}{(u_1-u_2+\frac{it_1}{N})(u_2-u_1+\frac{it_2}{N})}$$
$$= \frac{1}{N}[1 - \frac{1}{2}(t_1+t_2)^2 + \frac{1}{N}t_1t_2 + \frac{1}{12}(t_1+t_2)^4 - \frac{1}{2N}(t_1^3t_2 + t_1^2t_2^2 + t_1t_2^3)$$
$$+\frac{1}{6N^2}(t_1^3t^2 + t_1t_2^3) + \frac{1}{4N^2}t_1^2t_2^2 + O(t^6)] \tag{83}$$

Then

$$U_c(t_1,t_2,t_3) = \frac{2}{N^3} e^{-\frac{1}{2N}(t_1^2+t_2^2+t_3^2)}$$
$$\times \oint \prod \frac{du_j}{(2\pi i)} \frac{e^{\sum it_ju_j}\prod_{j=1}^3(1+\frac{it_j}{Nu_j})^N}{(u_1-u_2+\frac{it_1}{N})(u_2-u_3+\frac{it_2}{N})(u_3-u_1+\frac{it_3}{N})}$$
$$= \frac{2}{N^2}[1 - \frac{1}{2}(t_1+t_2+t_3)^2 + \frac{1}{12}(t_1+t_2+t_3)^4 + \frac{1}{N}(t_1t_2+t_1t_3+t_2t_3)$$
$$-\frac{1}{2N}(t_1^2t_2^2 + t_1^2t_3^2 + t_2^2t_3^2) - \frac{1}{2N}(t_1^3t_2 + t_1^3t_3 + t_1t_2^3 + t_2^3t^3 + t_1t_3^3 + t_2t_3^3)$$
$$-\frac{2}{N}t_1t_2t_3(t_1+t_2+t_3) + \frac{1}{24N^2}(t_1+t_2+t_3)^4$$
$$+\frac{1}{2N^2}t_1t_2t_3(t_1+t_2+t_3) + O(t^6)] \tag{84}$$

Combining these expansions one obtains

$$U_c(t_1+t_2+t_3) - NU(t_1+t_2,t_3) - NU(t_1+t_3,t_2) - NU(t_2+t_3,t_1)$$
$$+N^2U_c(t_1,t_2,t_3)$$

$$= \frac{1}{N^2}[\frac{1}{8}(t_1^4 + t_2^4 + t_3^4) + \frac{1}{4}(t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2) + t_1 t_2 t_3 (t_1 + t_2 + t_3)$$

$$+ \frac{1}{6}(t_1^3 t_2 + t_1^3 t_3 + t_2^3 t_1 + t_2^3 t_3 + t_3^2 t_1 + t_3^2 t_2) + O(t^6)] \tag{85}$$

It is order of $\frac{1}{N^2}$. Thus we see that the term, which we had considered in the large N limit, is cancelled by the additional terms terms in (85). In other words, the odd-power of $\frac{1}{N}$ are cancelled in the combination of (85). We thus have to expand also the denominator in (84), and compute the order $\frac{1}{N^4}$, instead of the order $\frac{1}{N^3}$ in $U_c(t_1, t_2, t_3)$.

Noting that expressions for the large N limit of $\frac{1}{N} < \mathrm{tr} M^{2k_1} \mathrm{tr} M^{2k_2} >$ is given by

$$\frac{1}{N^2} < \mathrm{tr} M^{2k_1} \mathrm{tr} M^{2k_2} > \sim \frac{\sqrt{k_1 k_2}}{\pi N^2 (k_1 + k_2)} 4^{k_1 + k_2}, \tag{86}$$

we have

$$\frac{1}{N^2} < \mathrm{tr} M^{2k_1 + 2k_2} \mathrm{tr} M^{2k_3} > + \frac{1}{N^2} < \mathrm{tr} M^{2k_1 + 2k_3} \mathrm{tr} M^{2k_2} >$$
$$+ \frac{1}{N^2} < \mathrm{tr} M^{2k_2 + 2k_3} \mathrm{tr} M^{2k_1} >$$
$$= - \frac{4^{k_1 + k_2 + k_3}}{\pi N^2 (k_1 + k_2 + k_3)} [\sqrt{k_3 (k_1 + k_2)} +$$
$$\sqrt{k_2 (k_1 + k_3)} + \sqrt{k_1 (k_2 + k_3)}] \tag{87}$$

This sum is identical to what we found in (81) for $\frac{2}{N^2} < \mathrm{tr} M^{2k_1} \mathrm{tr} M^{2k_2} \mathrm{tr} M^{2k_3} >$.

To discuss the next order terms, we first derive a formula for the correlation functions of the vertices, which is applicable to the general case of n-point vertex correlations for arbitrary genus. Let us use the following notation,

$$[k_i k_j] = \frac{k_i + k_j}{2N} \tag{88}$$

Returning to the expression for the correlation function of n-point vertices in the large k_i limit, one shifts $u_i \to u_1 - \frac{k_i}{2N}$,

$$U(t_1, ..., t_n) = e^{\frac{1}{12N^2} \sum k_j^3} \oint \frac{\prod du_i}{(2\pi i)^n} \frac{e^{\sum i t_i k_i + \frac{i t_i}{u_i}}}{\prod_{i,j} (u_i - u_j + [k_i k_j])}$$
$$= e^{\frac{1}{12N^2} \sum k_i^3} \oint \frac{du_i}{(2\pi i)^n} e^{\sum i t_i k_i + \frac{i t_i}{u_i}} (\sum_{\nu_1} \frac{(-[k_1 k_2])^{\nu_1}}{(u_1 - u_2)^{\nu_1 + 1}}) \cdots (\sum_{\nu_n} \frac{(-[k_n k_1])^{\nu_n}}{(u_n - u_1)^{\nu_n + 1}}) \tag{89}$$

From this representation one extracts $\frac{1}{N^n} < \prod_{j=1}^n \mathrm{tr} M^{2k_j} >_c$ as coefficient of the relevant power of t_i.

The above formula is applicable to the n-point case for arbitrary genus. For example, we consider n=3, and evaluate $\frac{1}{N^3} < \prod_{i=1}^{3} \mathrm{tr} M^{2k_i} >$.

We use the notation,

$$C_l(\nu) = \frac{(l+\nu)!}{l!\nu!} \tag{90}$$

with $C_l(0) = 1$. Expanding

$$\frac{1}{(u_1 - u_2)^{\nu+1}} = \frac{1}{u_1^{\nu+1}} \sum_{l=0}^{\infty} C_l(\nu) (\frac{u_2}{u_1})^l \tag{91}$$

the integral becomes

$$\begin{aligned}
I_{\nu_1,\nu_2,\nu_3} &= -\oint \frac{du_1 du_2 du_3}{(2\pi i)^3} \frac{(-1)^{\nu_1+\nu_2} [k_1 k_2]^{\nu_1} [k_2 k_3]^{\nu_2} [k_3 k_1]^{\nu_3}}{u_1^{\nu_1+\nu_3+2} u_2^{\nu_2+1}} \\
&\times \sum_{l_i, s_j} C_{l_1}(\nu_1) C_{l_2}(\nu_2) C_{l_3}(\nu_3) (\frac{u_2}{u_1})^{l_1} (\frac{u_3}{u_2})^{l_2} (\frac{u_3}{u_1})^{l_3} (iu_1)^{s_1} (iu_2)^{s_2} (iu_3)^{s_3} \\
& J_{s_1}(2t_1) J_{s_2}(2t_2) J_{s_3}(2t_3)
\end{aligned} \tag{92}$$

The u-integrals are now easy, and the coefficient of $(it_1)^{k_1}(it_2)^{k_2}(it_3)^{2k_3}$, denoted $I_{\nu_1,\nu_2,\nu_3}^{2k_1,2k_2,2k_3}$, is expressed as a sum. This sum is given by the contour integral

$$\begin{aligned}
& I_{\nu_1,\nu_2,\nu_3}^{2k_1,2k_2,2k_3} = -(-1)^{\nu_1+\nu_2} [k_1 k_2]^{\nu_1} [k_2 k_3]^{\nu_2} [k_3 k_1]^{\nu_3} \\
& \frac{1}{\prod (2k_i)!} \oint \frac{dx dy dz}{(2\pi i)^3} \sum (xy)^{\frac{l_1}{2}} (xz)^{\frac{l_3}{2}} (\frac{z}{y})^{\frac{l_2}{2}} C_{l_1}(\nu_1) C_{l_2}(\nu_2) C_{l_3}(\nu_3)
\end{aligned} \tag{93}$$

We consider separately (i) l_1, l_2 even, and l_3 odd, and (ii)l_1, l_3 even, and l_2 odd. When we consider the universal scaling limit for large k_i, this difference can be neglected. We replace $l_i \to 2l_i$ or $l_i \to 2l_i + 1$. The sum over l_i becomes

$$\sum_l C_{2l}(\nu)(xy)^l \sim \sum_l C_{2l+1}(\nu)(xy)^l \sim \frac{2^\nu}{(1-xy)^{\nu+1}} \tag{94}$$

For instance when $\nu = 3$, we have

$$\begin{aligned}
\sum_l C_{2l}(3)(xy)^l &= \frac{1}{6} \sum (2l+3)(2l+2)(2l+1)(xy)^l \\
&= \frac{1 + 6xy + (xy)^2}{(1-xy)^4}
\end{aligned} \tag{95}$$

and, since the saddle point is $x_c = y_c = 1$, the numerator at $x = y = 1$ is indeed equal to 2^3 .

Then one has

$$
\begin{aligned}
& I^{2k_1,2k_2,2k_3}_{\nu_1,\nu_2,\nu_3} = -(-1)^{\nu_1+\nu_2}[k_1k_2]^{\nu_1}[k_2k_3]^{\nu_2}[k_3k_1]^{\nu_3} \\
& \times\frac{1}{\prod(2k_i)!}\oint\frac{dxdydz}{(2\pi i)^3}\frac{(1+x)^{2k_1}(1+y)^{2k_2}(1+z)^{2k_3}2^{\nu_1+\nu_2+\nu_3}}{x^{k_1-\frac{\nu_1}{2}-\frac{\nu_3}{2}}y^{k_2+\frac{\nu_2}{2}+1}z^{k_3}(1-xy)^{\nu_1+1}(1-\frac{z}{y})^{\nu_2+1}(1-xz)^{\nu_3+1}} \\
& = \frac{1}{\prod(2k_i)!}\oint\frac{dxdydz}{(2\pi i)^3}\frac{(-1)^{\nu_3}(x+y)^{2k_1}(1+y)^{2k_2}(1+yz)^{2k_3}2^{\nu_1+\nu_2+\nu_3}}{x^{k_1}y^{k_1+k_2+k_3+1}z^{k_3}(x-1)^{\nu_1+1}(z-1)^{\nu_2+1}(xz-1)^{\nu_3+1}} \\
& \times[k_1k_2]^{\nu_1}[k_2k_3]^{\nu_2}[k_3k_1]^{\nu_3}
\end{aligned}
\tag{96}
$$

In the last line, we have changed $x \to \frac{x}{y}$ and $z \to zy$. We have also dropped the subleading powers of x and y for large k_i.

We have from (**??**),

$$
\frac{1}{N^n} < \prod_{j=1}^{3}\mathrm{tr}M^{2k_j} >_c = \prod_{j=1}^{3}(2k_j)!(\sum_{\nu} I^{2k_1,2k_2,2k_3}_{\nu_1,\nu_2,\nu_3})e^{\frac{1}{12N^2}\sum_{j=1}^{3}k_j^3}
\tag{97}
$$

As discussed before, we need the term of order of $\frac{1}{N^4}$, and the order $k^{3/2}$ in the large k limit. For this reason, we take $\nu_1+\nu_2+\nu_3=1$ in (96).

We have

$$
\frac{1}{(z-1)(xz-1)} = \frac{1}{z(x-1)}(\frac{1}{z-1}-\frac{1}{xz-1})
\tag{98}
$$

Using this identity, we have for $\nu_1+\nu_2+\nu_3=1$,

$$
\begin{aligned}
\frac{J}{N} &= \frac{[k_1k_2]}{(x-1)^2(z-1)(xz-1)} + \frac{[k_2k_3]}{(x-1)(z-1)^2(xz-1)} \\
&+ \frac{[k_3k_1]}{(x-1)(z-1)(xz-1)^2} \\
&= -\frac{k_3}{Ns_1^3s_3} + \frac{k_2+k_3}{2Ns_1^2s_3^2} + \frac{k_3}{Ns_1^3(s_1+s_3)} + \frac{k_3+k_1}{2Ns_1^2(s_1+s_3)^2}
\end{aligned}
\tag{99}
$$

where we have expanded x, y and z near the saddle points as $x = 1+is_1, y = 1+is_2, z = 1+is_3$. From the saddle point analysis, we obtain

$$
\begin{aligned}
& \frac{1}{N}\int\prod_{j=1}^{3}\frac{ds_j}{(2\pi i)}e^{-\frac{k_1}{4}s_1^2-\frac{k_1+k_2+k_3}{4}s_2^2-\frac{k_3}{4}s_3^2+\frac{k_1}{2}s_1s_2+\frac{k_3}{2}s_2s_3}J \\
& = I_1 + I_2
\end{aligned}
\tag{100}
$$

where I_1 corresponds to the first two terms of J, and I_2 corresponds to the third and fourth. For the s-integral, we integrate by parts, which reduces the integrand of J to the sum of constant terms and single pole terms in s_1 and

s_3. We take only the constant terms after diagonalization of the quadratic form in the exponent. We obtain

$$I_1 = \frac{\sqrt{k_1 k_2 k_3}(k_2 + k_3)}{8N\pi^{3/2}(k_1 + k_2 + k_3)}$$
$$I_2 = \frac{\sqrt{k_1 k_2 k_3}(k_1 + k_3)}{8N\pi^{3/2}(k_1 + k_2 + k_3)} \qquad (101)$$

Returning to the expansion of (92), we add the permutations over the k_i, and obtain

$$\frac{1}{N^3} < \mathrm{tr}M^{2k_1}\mathrm{tr}M^{2k_2}\mathrm{tr}M^{2k_3} >_c = \frac{4^{k_1+k_2+k_3}}{N^4}\frac{\sqrt{k_1 k_2 k_3}}{\pi^{3/2}} \qquad (102)$$

Dividing by by $\frac{4^{k_1+k_2+k_3}\sqrt{k_1 k_2 k_3}}{\pi^{3/2}}$, we obtain the intersection number

$$< \tau_0^3 >_{g=0} = 1 \qquad (103)$$

The intersection numbers, for the three point $n = 3$, for higher genuses are evaluated by considering higher values for ν_i in (97). After factoring out $\frac{4^{k_1+k_2+k_3}\sqrt{k_1 k_2 k_3}}{\pi^{3/2}}$, the vertex correlations $\frac{1}{N^3} < \mathrm{tr}M^{2k_1}\mathrm{tr}M^{2k_2}\mathrm{tr}M^{2k_3} >_c$ become polynomials in k_i, and scale as $\frac{k^3}{N^2}$. The intersection numbers are obtained from the coefficients of these polynomials.

5 Characteristic polynomials for Airy matrix functions and the replica method

We consider in this section the correlation functions for the characteristic polynomials of random matrices. The Airy matrix model of Kontsevich type, will then be derived from these correlation functions at the edge of the spectrum.

In a recent work we have studied the average of products of characteristic polynomials [7], defined as

$$\begin{aligned} F_k(\lambda_1, ..., \lambda_k) &= < \prod_{\alpha=1}^{k} \det(\lambda_\alpha - M) >_{A,M} \\ &= \int dM \prod_{i=1}^{k} \det(\lambda_i \cdot \mathrm{I} - \mathrm{M}) e^{-\frac{N}{2}\mathrm{trM}^2 + \mathrm{NtrMA}} \end{aligned} \qquad (104)$$

where M is an $N \times N$ Hermitian random matrix. It was shown that this correlation function has also a dual expression. This duality interchanges N, the size of the random matrix, with k, the number of points in F_k, as well as

the matrix source A with the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_k)$. Indeed we had derived that this same correlation function is given by [7]

$$F_k(\lambda_1, ..., \lambda_k) = \int dB \prod_{j=1}^{N} [\det(a_j - iB)] e^{-\frac{N}{2}\mathrm{trB^2+iNtrB\Lambda}} \tag{105}$$

where $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_k)$ and B is a $k \times k$ Hermitian matrix.

If we specialize this formula to a source A equal to the unit matrix, providing thus a trivial constant shift for M, the formula (105) involves

$$\begin{aligned} \det(1 - iB)^N &= \exp[N \mathrm{trln}(1 - iB)] \\ &= \exp[-iN\mathrm{tr}B + \frac{N}{2}\mathrm{tr}B^2 + i\frac{N}{3}\mathrm{tr}B^3 + \cdots] \end{aligned} \tag{106}$$

The linear term in B in (106), combined with the linear term of the exponent of (105), shifts Λ by one. The B^2 terms in (105) cancel. In a scale in which the initials λ_k are close to one, or more precisely $N^{2/3}(\lambda_k - 1)$ is finite, the large N asymptotics of (105) is given by matrices B of order $N^{-1/3}$. Then the higher terms in (106) are negligible and we are left with terms linear and cubic in the exponent, namely

$$F_k(\lambda_1, ..., \lambda_k) = \int dB \prod_{j=1}^{N} e^{i\frac{N}{3}\mathrm{tr}B^3 + iN\mathrm{tr}B(\Lambda - 1)}. \tag{107}$$

This is nearly identical to the matrix Airy integral, namely Kontsevich's model [15], which gives the intersection numbers of moduli of curves.

The original Kontsevich partition function was defined as

$$Z = \frac{1}{Z'} \int dM e^{-\frac{1}{2}\mathrm{tr\Lambda M^2} + \frac{i}{6}\mathrm{trM^3}} \tag{108}$$

where $Z' = \int dM e^{-\frac{1}{2}\mathrm{trM^2}}$. The shift $M \to M - i\Lambda$, eliminates the M^2 term and one recovers (107).

Let us examine the simple case of a 2×2 matrix M, with two points λ_1 and λ_2. (Indeed this simple $N = 2$ case is useful as a check for the intersection numbers. For higher N one could perform a similar analysis.) Then we have

$$Z = \sqrt{\lambda_1\lambda_2}(\lambda_1 + \lambda_2) e^{\frac{1}{2}(\lambda_1^3 + \lambda_2^3)} Y \tag{109}$$

where

$$\begin{aligned} Y &= \int dM e^{\frac{i}{2}\mathrm{tr\Lambda^2 M} + \frac{i}{6}\mathrm{trM^3}} \\ &= \int dx_1 dx_2 (\frac{x_1 - x_2}{\lambda_1^2 - \lambda_2^2}) e^{\frac{i}{2}(\lambda_1^2 x_1 + \lambda_2^2 x_2) + \frac{i}{6}(x_1^3 + x_2^3)} \\ &= \frac{2}{\lambda_1^2 - \lambda_2^2}(\frac{\partial}{\partial\lambda_1^2} - \frac{\partial}{\partial\lambda_2^2})[\frac{1}{\sqrt{\lambda_1\lambda_2}} e^{-\frac{1}{3}(\lambda_1^3 + \lambda_2^3)} z(\lambda_1) z(\lambda_2)] \end{aligned} \tag{110}$$

where

$$z(\lambda)=\frac{\int dx e^{-\frac{x^2}{2}\lambda+\frac{i}{6}x^3}}{\int e^{-\frac{x^2}{2}\lambda}dx} \tag{111}$$

Then we get

$$Z=1+\frac{1}{6}\tilde{t}_0^3+\frac{1}{24}\tilde{t}_1+O(\frac{1}{\lambda^5}), \tag{112}$$

$$\begin{aligned}\log Z &= \frac{1}{6}\tilde{t}_0^3+\frac{1}{24}\tilde{t}_1+\frac{1}{6}\tilde{t}_0^3\tilde{t}_1+\frac{1}{48}\tilde{t}_1^2+\frac{1}{24}\tilde{t}_0\tilde{t}_2\\ &+\frac{1}{6}\tilde{t}_1^2\tilde{t}_0^3+\frac{1}{72}\tilde{t}_1^3+\frac{1}{48}\tilde{t}_3t_0^2+\frac{1}{12}\tilde{t}_0\tilde{t}_1\tilde{t}_2\\ &+\frac{1}{1152}\tilde{t}_4+O(\frac{1}{\lambda^{12}})\end{aligned} \tag{113}$$

where we have defined the moduli parameter $\tilde{t}_i$ as

$$\tilde{t}_0=\frac{1}{\lambda_1}+\frac{1}{\lambda_2},\quad \tilde{t}_1=\frac{1}{\lambda_1^3}+\frac{1}{\lambda_2^3},\quad \tilde{t}_n=\frac{1}{\lambda_1^{2n+1}}+\frac{1}{\lambda_2^{2n+1}}. \tag{114}$$

(We need the notation $\tilde{t}_n$, instead of the usual notation t_n, for distinguishing those parameters from the Fourier transform parameters t_i).

From these coefficients, we recover the known results

$$\begin{aligned}&<\tau_0^3>_{g=0}=1,\quad <\tau_1>_{g=1}=\frac{1}{24},\quad <\tau_0^3\tau_1>_{g=0}=1,\\ &<\tau_1^2>_{g=1}=\frac{1}{24},\quad <\tau_0\tau_2>_{g=1}=\frac{1}{24}\\ &<\tau_0^3\tau_1^2>_{g=0}=2,\quad <\tau_1^3>_{g=1}=\frac{1}{12},\quad <\tau_0^2\tau_3>_{g=1}=\frac{1}{24}\\ &<\tau_0\tau_1\tau_2>g=1=\frac{1}{12},\quad <\tau_4>_{g=2}=\frac{1}{1152}\end{aligned} \tag{115}$$

The generating function for the intersection numbers $<\prod\tau_i^{d_i}>_g$ is

$$\log Z=\sum_{m_l}<\tau_0^{m_1}\tau_1^{m_2}\cdots>\prod_{l=0}^{\infty}\frac{\tilde{t}_l^{m_l}}{m_l!} \tag{116}$$

where $\tilde{t}_l$ is related to the eigenvalues of the matrix Λ as

$$\tilde{t}_l=\sum_{j=1}^{N}\frac{(2l-1)!!}{\lambda_j^{2l+1}} \tag{117}$$

Note that the generating function of the intersection numbers (116) is quite different from the generating function (67), which was obtained from the correlation functions of the vertices. The relation between $\tilde{t}_l$ and x_j are

$$\tilde{t}_l\sim x_j^l \tag{118}$$

where $x_j = \frac{k_j}{2^{1/3}}$, in which k_j is the power in M^{2k_j}. However the degenerate case needs some precaution, when we have $\tilde{t}_l^2 \sim x_j^l x_m^l$ for $j \neq m$.

The replica method for the correlation functions has been used earlier in random matrix theory for the GUE [8, 9]. Following this replica analysis we study now two types of correlation functions. The first one is the correlation function for the eigenvalues of the matrix M :

$$\rho(\lambda_1, ..., \lambda_k) =< \prod_{\alpha=1}^{k} \frac{1}{N} \mathrm{tr}\delta(\lambda_\alpha - M) >_{A,M} \tag{119}$$

The second type of correlation functions is the average of products of characteristic polynomials [7],

$$F_k(\lambda_1, ..., \lambda_k) =< \prod_{\alpha=1}^{k} \det(\lambda_\alpha - M) >_{A,M} . \tag{120}$$

where the average $< ... >_{A,M}$ is with respect to the probability distribution $P_A(M)$,

$$P_A(M) = \frac{1}{Z} e^{-\frac{N}{2}\mathrm{tr}M^2 + N\mathrm{tr}MA} \tag{121}$$

The random matrix M is a complex Hermitian N by N matrix and A is an external source, which we can take as a diagonal matrix, $A = \mathrm{diag}(a_1, ..., a_N)$ since the integration measure is unitary invariant. If A is zero, it reduces to the Gaussian unitary ensemble (GUE), but it is convenient to use the probability distribution $P_A(M)$ for setting up the replica method, even if we let $A = 0$ at the end. The correlation functions with the distribution $P_A(M)$ require the HarishChandra-Itzykson-Zuber formula [11, 12] for the unitary group integral.

We have evaluated the F.T. of the correlation functions near the edge in the previous sections. To prove that F.T. of the correlation function is the generating function of the intersection numbers, we express these functions as the zero replica limit of characteristic polynomials average. Let us begin with the one point function, namely the density of states ; we first use the identity

$$< \mathrm{tr}\delta(\lambda - M) >_{A,M} = \frac{1}{\pi} \Im\mathrm{m} \lim_{n\to 0} \frac{1}{n} \frac{\partial}{\partial \lambda} < [\det(\lambda - i\epsilon - M)]^n >_{A,M} \tag{122}$$

and use the duality derived in [7] to write

$$< [\det(\lambda - M)]^n >_{A,M} =< \prod_{\gamma=1}^{N} [\det(a_\gamma - iB)] >_{\Lambda,B} \tag{123}$$

where B is an $n \times n$ random Hermitian matrix, and Λ, in this case, is a multiple of the $n \times n$ identity matrix : $\Lambda = \mathrm{diag}(\lambda, ..., \lambda)$. Note that we have

traded an n-point function of $N \times N$ matrices M for an N point function of $n \times n$ matrices B.

If we explore the edge of the distribution by taking again $a_\gamma = 1$, and expand in powers of B, we find that the B^2 term cancels, and obtain

$$U(t) =< \frac{1}{N}\mathrm{tr}\, e^{itM} >= \Im\mathrm{m} \lim_{n\to 0} \int d\lambda e^{it\lambda} \frac{1}{n}\frac{\partial}{\partial\lambda} \int dB e^{\mathrm{tr}[i\frac{N}{2}B^3 + iB(\lambda-1)]} \quad (124)$$

After integration by parts over λ we obtain

$$U(t) = t \lim_{\mathrm{n}\to 0} \int \mathrm{d}\lambda \mathrm{e}^{\mathrm{it}\lambda} \frac{1}{\mathrm{n}} \int \mathrm{dBe}^{\mathrm{itr}[\frac{\mathrm{N}}{2}\mathrm{B}^3 + \mathrm{iB}(\lambda-1)]} \quad (125)$$

Since the replica parameter n means the repetition n times of the same λ , we simply replace the $\tilde{t}_l$ of the previous un-replicated case by

$$\tilde{t}_l = \sum_{j=1}^{k} \frac{1}{\lambda_j^l} \to \sum_{j=1}^{k} \frac{n}{\lambda_j^l} \quad (126)$$

For the one point function k=1, the expansion of the Airy matrix model in terms of $\tilde{t}_l$ is thus a power series in n. In the zero-replica limit one can neglect all terms beyond the linear one in n. Thus, $U(t)$ in (125) is expressed as linear combinations of the $\tilde{t}_l$. The F.T. of $\tilde{t}_l = \frac{1}{\lambda^l}$ yields dimensionally a factor t^l. Therefore, $U(t)$ appears as a power series in t, whose coefficients are the intersection numbers $< \tau_l >$, the same coefficients obtained in the Kontsevich model.

This argument holds also for the two point correlation function, since

$$\lim_{n\to 0} \frac{\partial^2}{\partial\lambda_1\partial\lambda_2} \frac{1}{n^2} < [\det(\lambda_1 - M)\det(\lambda_2 - M)]^n >= \mathrm{tr}\frac{1}{\lambda_1 - \mathrm{M}}\mathrm{tr}\frac{1}{\lambda_2 - \mathrm{M}} \quad (127)$$

In this two-point case (k=2), one deals with λ_1 and λ_2 and $\tilde{t}_l$ becomes

$$\tilde{t}_l = \frac{n}{\lambda_1^l} + \frac{n}{\lambda_2^l} \quad (128)$$

The replica limit ($n \to 0$) requires to retain the terms of order n^2 , i.e. products of the form $\tilde{t}_l\tilde{t}_m$. Then $U(t_1, t_2)$ is a power series in t_1, t_2, whose coefficients are the intersection numbers $< \tau_l\tau_m >$.

The F.T. of the k-point correlation function $U(t_1, ..., t_k)$ produces, by the same replica method, the intersection numbers $< \tau_{m_1}\tau_{m_2}... >$ as coefficients of $t_1^{m_1}t_2^{m_2}\cdots t_k^{m_k}$.

Thus we have shown that the F.T. of the correlation functions in the edge region, is the generating function of the intersection numbers , our main conclusion.

6 Summary

We have derived the expressions for the correlation functions of the vertices $< \frac{1}{N^n} \prod_{i=1}^n \mathrm{tr} M^{2k_i} >$, when N and k_i are simultaneously large, in a scaling region in which k^3 scales like N^2. The expressions are obtained from an exact contour integral representation valid for finite N and for an arbitrary external matrix source. The coefficients of the expansion of the correlation functions provide the intersection numbers of the moduli space of curves.

The correlation functions of the eigenvalues may also be obtained from the average of characteristic polynomials in a zero-replica limit. Using a previously derived duality in which the size N of the matrix is interchanged with the number of points of the correlation functions, we have recovered the Airy matrix model of Kontsevich and re-derived the intersection numbers by a simple saddle-point analysis.

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